Lecture 4. Linear First-Order Equations

In this lecture notes, we will talk about

- Classification of Differential Equations
  - Order of the equation
  - Linear/non-linear equations
- Linear First-Order Equations
  - Integrating Factor Method

Classification of Differential Equations

Order: The **order** of a differential equation is the order of the highest derivative that appears in the equation.

Linear Differential Equation: A **linear differential equation** is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives, that is,

$$a_0(x)y + a_1(x)y' + a_2(x)y'' \cdots + a_n(x)y^{(n)} = b(x)$$

where $a_0(x), \ldots, a_n(x)$ and $b(x)$ are arbitrary differentiable functions that do not need to be linear, and $y', \ldots, y^{(n)}$ are the successive derivatives of an unknown function y of the variable x.

Remark. Linearity is important because the structure of the family of solutions to a linear equation is relatively simple. Linear equations can usually be explicitly solved.

Eg: $y^2 + y' = 0$ not linear.

$\sqrt{y'} + xy'' = 0$ not linear.

Example 1. Determine the order of the given differential equation and state whether the equation is linear or nonlinear.

1. $(1-x)y'' - 4xy' + 5y = \cos(x)$

   Linear 2nd

2. $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + \left(\tan^2(t)\right)y = t^5$

   linean 3rd

3. $y'' - y + y^2 = 0$

   Nonlinear 2nd

4. $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = 1$

   Linear 4th order

5. $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

   Nonlinear. 2nd

Exercise 2. Determine whether each first-order differential equation is separable, linear, both, or neither.

1. $\dfrac{dy}{dx} + e^x y = x^2 y^2$

   Neither

2. $y + \cot x = x^5 y'$

   Linear

3. $y \ln x - x^2 y = xy'$

   Both

4. $\dfrac{dy}{dx} + e^y = \tan x$

   Neither

An example. Find a general solution to the differential equation

$$\frac{dy}{dx} = 2xy \quad (y > 0)$$

ANS : Method 1 Notice that this is a separable diff eqn. So

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\Rightarrow \boxed{\ln y = x^2 + C}$$

Method 2 Rather than dividing both sides by y, we can multiply both sides of (1) by $\frac{1}{y}$.

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x \qquad D_x(\ln y) = \frac{1}{y} \cdot \frac{dy}{dx} \text{ (chain rule)}$$

We can recognize each side of the eqn as a derivative. i.e.

$$D_x(\ln y) = D_x(x^2)$$

Then integrating both sides gives us

$$\boxed{\ln y = x^2 + c}$$

In general, an integrating factor for a diff. eqn. is a function $\rho(x,y)$ with the property that multiplying each side of the eqn by $\rho(x,y)$ allows each side to be recognizable as derivative. For example, $\rho(x,y) = \frac{1}{y}$ is an integrating factor for this example.

Linear First-order Equations

A **linear first-order equation** is a differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where the coefficient functions $P(x)$ and $Q(x)$ are continuous on some interval on the x-axis.

- This equation can always be solved using the integrating factor

$$\rho(x) = e^{\int P(x)dx}$$

Rmk: No constant of the integration is needed when finding the integrating factor $\rho(x)$.
since replacing $\int P(x)\,dx$ with $\int P(x)dx + c$ leads to $\rho(x) = e^{c + \int P(x)dx} = e^{c} \cdot e^{\int P(x)dx}$

- Multiplying by $\rho(x)$ gives

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

- The left-hand side is now the derivative of the product

$$y(x) \cdot e^{\int P(x)dx}$$

Check: $D_x\left(y(x) \cdot e^{\int P(x)\,dx}\right) = \frac{dy}{dx} \cdot e^{\int P(x)dx} + y \cdot e^{\int P(x)dx} \cdot P(x)$

(product rule + chain rule)

- So we can rewrite our equation as

$$D_x\left[y(x) \cdot e^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx}$$

- Integrating both sides gives

$$y(x)e^{\int P(x)dx} = \int \left(Q(x)e^{\int P(x)dx}\right)dx + C$$

- Finally, solving for $y(x)$ gives

$$y(x) = e^{-\int P(x)dx}\left[\int \left(Q(x)e^{\int P(x)dx}\right)dx + C\right]$$

- **Note:** This formula is not to be memorized, but rather illustrates a general method that can be applied in specific cases.

We summarize the steps of the method as follows:

Method of Solution of Linear First-Order Equations

Rmk: We need to make sure we have the eqn in this form. See example 2 below

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Step 1. Compute the integrating factor $\rho(x) = e^{\int P(x)dx}$.

Step 2. Multiply both sides of the differential equation by $\rho(x)$.

Step 3. Next, recognize the left-hand side of the resulting equation as the derivative of a product:

$$D_x[\rho(x)y(x)] = \rho(x)Q(x)$$

Step 4. Finally, integrate this equation,

$$\rho(x)y(x) = \int \rho(x)Q(x)dx + C$$

then solve for $y(x)$ to obtain the general solution of the original differential equation.

Example 3. Find a general solution to the differential equation

$$xy' = 3y + x^4 \cos x, \quad y(2\pi) = 0$$

ANS: We first write (12) in the form of $\frac{dy}{dx} + P(x)\, y = Q(x)$.

If $x \neq 0$, we can rewrite (12) as

$$\frac{dy}{dx} \underbrace{- \frac{3}{x}}_{P(x)} y = \underbrace{x^3 \cos x}_{Q(x)} \qquad \circledast$$

Step 1. An integrating factor $\rho(x) = e^{\int P(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|}$

$= |x|^{-3}$. If $x > 0$, $\underline{\rho(x) = x^{-3}}$. If $x < -3$, $\rho(x) = -x^{-3}$

Step 2. Multiply both sides by $\rho(x) = x^{-3}$

$$x^{-3}\frac{dy}{dx} - 3x^{-4}y = \cos x$$

Step 3. Note $LHS = D_x(x^{-3}y)$ $(= D_x(\rho(x)y(x)))$

Step 4. Integrate both sides in terms of x.

$$x^{-3}y = \int \cos x \, dx + C = \sin x + C$$

$$\Rightarrow y = x^3 \sin x + x^3 \cdot C$$

As $y(2\pi) = 0$, $(2\pi)^3 \cdot \cancel{\sin 2\pi}^{\,0} + (2\pi)^3 \cdot C = 0 \Rightarrow C = 0$

$$\boxed{y = x^3 \sin x}$$

Example 4. Solve the following initial value problem:

$$t\frac{dy}{dt} + 3y = 5t$$

with $y(1) = 2$.

ANS: This is a linear first order eqn.

We rewrite the eqn:

$$\frac{dy}{dt} + \underbrace{\frac{3}{t}}_{P(t)} y = \underbrace{5}_{Q(t)}.$$

Compute the integrating factor

$$\rho(t) = e^{\int \frac{3}{t} dt} = e^{3\ln|t|} = e^{\ln|t|^3} = |t|^3$$

So we can take $\rho(t) = t^3$

Multiply both sides of the eqn by $\rho(t) = t^3$.

We know

$$\underbrace{D_t}_{\frac{\partial}{\partial t}}(\rho(t)\, y(t)) = 5 \cdot \rho(t) = 5t^3$$

Integrate both sides, we get

$$\rho(t) y(t) = t^3 y(t) = \int 5t^3 dt + C = \frac{5}{4}t^4 + C$$

$$\Rightarrow y(t) = \frac{5}{4}t + \frac{C}{t^3}$$

As $y(1) = 2$, $y(1) = \frac{5}{4} \cdot 1 + \frac{C}{1^3} = 2 \Rightarrow C = \frac{3}{4}$

Thus $y(t) = \frac{5}{4}t + \frac{3}{4}t^{-3}$

Exercise 5. Solve the initial value problem

$$y' + \frac{1}{x+4} y = x^{-2}, \quad y(1) = 7$$

Solution. This is a linear first-order equation in the standard form. We compute hte integral factor"

$$\rho(x) = e^{\int \frac{1}{x+4} dx} = e^{\ln(x+4)} = x + 4$$

Multiplying the both sides of the equation by $\rho(x) = x + 4$ gives

$$(x+4)y' + y = x^{-1} + 4x^{-2}$$

Integration on both sides, we have

$$\rho(x)y = (x+4)y = \ln x - 4x^{-1} + C$$

Thus we have

$$y(x) = \frac{1}{x+4}\left(C + \ln x - \frac{4}{x}\right)$$

The initial condition $y(1) = 7$ allows us to determine the value of C :

$$7 = \frac{1}{5}(C - 4) \text{ so } C = 39.$$

The solution to the initial value problem is therefore

$$y(x) = \frac{1}{x+4}\left(39 + \ln(x) - \frac{4}{x}\right).$$

Exercise 6. Solve the initial value problem.

$$\frac{dy}{dx} - 2xy = 8x, \quad y(0) = 0$$

Solution. This is a linear first-order equation in the standard form. We compute the integral factor

$$\rho(x) = e^{\int -2x dx} = e^{-x^2}$$

Multiplying the both sides of the equation by e^{-x^2} gives

$$e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = 8xe^{-x^2}$$

integrating we get

$$\rho(x)y(x) = e^{-x^2}y = \int 8xe^{-x^2}dx = -\frac{8}{2}\int e^{-x^2}d(-x^2) = -4e^{-x^2} + C$$

finally solving for $y(x)$ gives

$$y(x) = Ce^{x^2} - 4$$

Using $y(1) = 0$ we get $C = 4$, giving

$$y = 4e^{x^2} - 4$$